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LETTER TO THE EDITOR

Algebraic Bethe ansatz approach for the one-dimensional Hubbard model

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Abstract. We formulate in terms of the quantum inverse scattering method the algebraic Bethe ansatz solution of the one-dimensional Hubbard model. The method developed is based on a new set of commutation relations which encodes a hidden symmetry of six-vertex type.

In 1968 the exact solution of the one-dimensional Hubbard model by the coordinate Bethe ansatz was presented by Lieb and Wu [1, 2]. It took some years until Shastry found the many conserved charges [3] and also the two-dimensional classical vertex model [4, 5] whose transfer matrix commutes with the Hubbard Hamiltonian. The R-matrix responsible for the integrability ('infinite number of conserved charges') was then explicitly exhibited [4, 5]. Some time later, Wadati *et al* [6] were able to verify such results by using a quite different and interesting approach. Afterwards, Bariev [7] developed a variant of the coordinate Bethe ansatz to study Shastry's vertex model, although on the basis of a diagonal-to-diagonal transfer matrix approach [8]. More recently, some progress has been made concerning the Yangian symmetry of the Hubbard model [9] and also on its 'free-fermion' Yang–Baxter structure [10]. Some further discussion about Hubbard's invariant can also be found in the literature [11].

However, certain important properties underlying such an 'integrable' programme still needs to be understood. This is justified, for example, by the early attempt of Shastry [5] in conjecturing the eigenvalues of the row-to-row transfer matrix of the 'covering' vertex model. An important step towards closing this program is certainly the formulation of the Bethe states of the one-dimensional Hubbard model by means of the quantum inverse scattering approach [13]. Unlike the standard Bethe ansatz, this method is based on first-principle algebraic rules and definitely brings new insight on the mathematical structure of integrable systems. The solution of the Hubbard model by the quantum inverse scattering method is, in fact, a long-standing problem in the field of exactly solved models. In this letter we show how this more unified approach of Bethe ansatz technique can be established for the one-dimensional Hubbard model. In the course of our formulation we had to overcome a major difficulty: the non-additive property of the Hubbard R-matrix. We have found the fundamental commutation rules between the creation and annihilation operators present in the embedding vertex model. It turns out that the eigenvectors, eigenvalues and the Bethe ansatz equations follow as a consequence of systematic algebraic manipulation of such commutation rules. A hidden symmetry of six-vertex type, important for integrability, is noted. We think that our results should also be of relevance for future developments

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of the physical properties of the one-dimensional Hubbard model. One possibility should be the application of our formulation in the context of the Korepin *et al* method [15] of computing correlation functions.

The Hamiltonian of the Hubbard model on a one-dimensional lattice of length L is written in the form

$$H = -\sum_{i=1}^{L} \sum_{\sigma=\uparrow,\downarrow} (c_{i,\sigma}^{\dagger} c_{i+1,\sigma} + \mathrm{HC}) + U \sum_{i=1}^{L} n_{i,\uparrow} n_{i,\downarrow} \tag{1}$$

where $c_{i,\sigma}(n_{i,\sigma})$ are Fermi (number) operators with spin σ on site i, and U is the Hubbard coupling. Fundamental to the integrability of the Hubbard model is the fact that Hamiltonian (1) commutes with a certain one-parameter family of the transfer matrix T. In analogy with integrable systems in classical mechanics, T is the generator of the many conserved charges, and Hamiltonian (1) is one of those currents. Thus, the analysis of the physical properties of the transfer matrix will certainly provide a deeper understanding of the Hubbard model. The appropriate two-dimensional classical statistical system exhibiting such properties was found by Shastry [4, 5]. The model is constituted of two coupled six-vertex models satisfying the free-fermion condition. The vertex model is parametrized in terms of three functions, $a(\lambda)$, $b(\lambda)$ and $h(\lambda)$, which are constrained by Hubbard's coupling as

$$\sinh[2h(\lambda)] = \tfrac{1}{2} U a(\lambda) b(\lambda) \tag{2}$$

where functions $a(\lambda)$ and $b(\lambda)$ are the non-trivial free-fermion Boltzmann weights. This gives us an R-matrix $R(\lambda, \mu)$ consisting of ten distinct Boltzmann weights. Here we denote them by $\alpha_i(\lambda, \mu)$, $i = 1, \ldots, 10$. In the appendix we present the structure of the R-matrix, the explicit expressions and some useful identities for the weights $\alpha_i(\lambda, \mu)$ [4, 6]. In general, the transfer matrix T is obtained as a trace of an auxiliary monodromy operator, $T = \mathrm{Tr}_G\, \mathcal{T}$. The space G is a 'ghost' variable, corresponding to a horizontal arrow in the classical vertex model. Its dimension corresponds to the four possible states of the Hubbard model on a given site. As we shall see below, it is convenient to write the associated monodromy matrix $\mathcal{T}(\lambda)$ as

$$\mathcal{T}(\lambda) = \begin{pmatrix} B(\lambda) & \boldsymbol{B}(\lambda) & F(\lambda) \\ \boldsymbol{C}(\lambda) & \hat{A}(\lambda) & \boldsymbol{B}^*(\lambda) \\ C(\lambda) & \boldsymbol{C}^*(\lambda) & D(\lambda) \end{pmatrix} \tag{3}$$

where $\boldsymbol{B}(\lambda)$ $(\boldsymbol{B}^*(\lambda))$ and $\boldsymbol{C}(\lambda)$ $(\boldsymbol{C}^*(\lambda))$ are two component vectors with dimensions $1 \times 2(2 \times 1)$ and $2 \times 1(1 \times 2)$, respectively. The operator $\hat{A}(\lambda)$ is a 2×2 matrix and the other remaining operators are scalars. The integrability condition is based on the Yang–Baxter algebra, namely

$$R(\lambda, \mu)\mathcal{T}(\lambda) \overset{s}{\otimes} \mathcal{T}(\mu) = \mathcal{T}(\mu) \overset{s}{\otimes} \mathcal{T}(\lambda) R(\lambda, \mu) \tag{4}$$

where the symbol $\overset{s}{\otimes}$ stands for the Grassmann direct product [12]. Such a definition takes into account the extra signs appearing when fermionic states (spin up and down) are permuted [6]. One consequence of Shastry's Boltzmann weights is that the monodromy matrix has a triangular form when acting on the standard ferromagnetic pseudovacuum $|0\rangle$. More precisely, we find the following diagonal properties

$$B(\lambda)\,|0\rangle = \left[\frac{a(\lambda)}{b(\lambda)}\,\mathrm{e}^{2h(\lambda)}\right]^L |0\rangle \qquad D(\lambda)\,|0\rangle = \left[\frac{b(\lambda)}{a(\lambda)}\,\mathrm{e}^{2h(\lambda)}\right]^L |0\rangle$$
$$A_{aa}(\lambda)\,|0\rangle = |0\rangle \qquad a = 1, 2 \tag{5}$$

as well as the annihilation identities

$$C(\lambda)\,|0\rangle = \boldsymbol{C}(\lambda)\,|0\rangle = \boldsymbol{C}^*(\lambda)\,|0\rangle = 0 \qquad A_{ab}(\lambda)\,|0\rangle = 0\ (a \neq b = 1, 2). \tag{6}$$

This suggests that operators $\boldsymbol{B}(\lambda)$, $\boldsymbol{B}^*(\lambda)$ and $F(\lambda)$ act as creator fields on the ferromagnetic reference state $|0\rangle$. We notice, however, that the operators $\boldsymbol{B}(\lambda)$ and $\boldsymbol{B}^*(\lambda)$ do not mix under the integrability condition (4). Therefore, in the construction of the eigenvectors it will be enough to look only for combinations between the fields $\boldsymbol{B}(\lambda)$ and $F(\lambda)$. The one-particle state $|\Phi_1(\lambda_1)\rangle$ is made by the linear combination

$$|\Phi_1(\lambda_1)\rangle = \boldsymbol{B}(\lambda_1)\cdot\vec{\mathcal{F}}\,|0\rangle = B_a(\lambda_1)\mathcal{F}^a\,|0\rangle \tag{7}$$

where $\mathcal{F}^a$ is the component of a constant vector $\vec{\mathcal{F}}$ with dimension (2×1). The two-particle state $|\Phi_2(\lambda_1,\lambda_2)\rangle$ depends both of operators $\boldsymbol{B}(\lambda)$ and $F(\lambda)$. This happens because the commutation rule between the two fields of type $\boldsymbol{B}(\lambda)$ generates the scalar operator $F(\lambda)$. This is a constraint imposed by the integrability condition (4), which reads

$$\begin{aligned}\boldsymbol{B}(\lambda)\otimes\boldsymbol{B}(\mu) &= \alpha_{1,2}(\lambda,\mu)[\boldsymbol{B}(\mu)\otimes\boldsymbol{B}(\lambda)]\cdot\hat{r}(\lambda,\mu)\\ &\quad -\mathrm{i}\alpha_{10,7}(\lambda,\mu)\{F(\lambda)B(\mu)-F(\mu)B(\lambda)\}\boldsymbol{\xi}\end{aligned} \tag{8}$$

where we define $\alpha_{a,b}(\lambda,\mu) = \alpha_a(\lambda,\mu)/\alpha_b(\lambda,\mu)$. The vector $\boldsymbol{\xi}$ and the matrix $\hat{r}(\lambda,\mu)$ have the following structures

$$\boldsymbol{\xi} = \begin{pmatrix}0 & 1 & -1 & 0\end{pmatrix} \qquad \hat{r}(\lambda,\mu) = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & a(\lambda,\mu) & b(\lambda,\mu) & 0\\ 0 & b(\lambda,\mu) & a(\lambda,\mu) & 0\\ 0 & 0 & 0 & 1\end{pmatrix} \tag{9}$$

where

$$b(\lambda,\mu) = \alpha_{8,1}(\lambda,\mu)\alpha_{9,7}(\lambda,\mu) \qquad a(\lambda,\mu)+b(\lambda,\mu) = 1. \tag{10}$$

Remarkable enough we have found that the $\hat{r}$-matrix (9) is in fact factorizable. Moreover, when properly parametrized, it has the same structure as that appearing in the isotropic six-vertex model. We stress that such hidden symmetry is crucial in our algebraic construction and plays a fundamental role on the exact solution of the Hubbard model. In our opinion, this is the 'nice' algebraic explanation for the fact that the bare two-body scattering of the Hubbard Hamiltonian appears in the six-vertex form [1, 2]. This result can be established by performing the following change of variables:

$$\tilde{\lambda} = \frac{a(\lambda)}{b(\lambda)}\mathrm{e}^{2h(\lambda)} - \frac{b(\lambda)}{a(\lambda)}\mathrm{e}^{-2h(\lambda)} - \frac{U}{2}. \tag{11}$$

By using the Boltzmann weights [4, 6] (see the appendix) in equation (10) and by considering the new variables defined in (11), we are able to rewrite functions $a(\tilde{\lambda},\tilde{\mu})$ and $b(\tilde{\lambda},\tilde{\mu})$ as

$$a(\tilde{\lambda},\tilde{\mu}) = \frac{U}{\tilde{\mu}-\tilde{\lambda}+U} \qquad b(\tilde{\lambda},\tilde{\mu}) = \frac{\tilde{\mu}-\tilde{\lambda}}{\tilde{\mu}-\tilde{\lambda}+U} \tag{12}$$

which are precisely the non-trivial Boltzmann weights of the isotropic six-vertex model [13–15]. Taking into account our considerations above, it is not difficult to check that the two-particle state is given by

$$|\Phi_2(\lambda_1,\lambda_2)\rangle = \{\boldsymbol{B}(\lambda_1)\otimes\boldsymbol{B}(\lambda_2) + \mathrm{i}\alpha_{10,7}(\lambda_1,\lambda_2)F(\lambda_1)(\boldsymbol{\xi}\otimes\boldsymbol{\Phi}_0)B(\lambda_2)\}\cdot\vec{\mathcal{F}}\,|0\rangle \tag{13}$$

where $\boldsymbol{\Phi}_0$ is the unitary constant. In fact, we have checked that all unwanted terms generated by the eigenvalue problem can be cancelled out through a unique Bethe ansatz equation. Moreover, at least at this level, the physical meaning of our construction is the following.

While each component of the field $\boldsymbol{B}(\lambda)$ creates an electron with spin up or down, the operator $F(\lambda)$ is responsible for the double occupancy on a given site of the lattice. In general, the n-particle state can be constructed by induction and we have verified that it satisfies the following recurrence relation

$$|\Phi_n(\lambda_1, \ldots, \lambda_n)\rangle = \boldsymbol{\Phi}_n(\lambda_1, \ldots, \lambda_n) \cdot \vec{\mathcal{F}}\,|0\rangle \tag{14}$$

where

$$\begin{aligned}\boldsymbol{\Phi}_n(\lambda_1, \ldots, \lambda_n) = \boldsymbol{B}(\lambda_1) \otimes \boldsymbol{\Phi}_{n-1}(\lambda_2, \ldots, \lambda_n) + \sum_{j=2}^{n} \mathrm{i}\alpha_{10,7}(\lambda_1, \lambda_j) \\ \times \prod_{k=2, k\neq j}^{n} \mathrm{i}\alpha_{2,9}(\lambda_k, \lambda_j)[\boldsymbol{\xi} \otimes F(\lambda_1)\boldsymbol{\Phi}_{n-2}(\lambda_2, \ldots, \lambda_{j-1}, \lambda_{j+1}, \ldots, \lambda_n)B(\lambda_j)] \\ \times \prod_{k=2}^{j-1} \alpha_{1,2}(\lambda_k, \lambda_j)\hat{r}_{k,k+1}(\lambda_k, \lambda_j).\end{aligned} \tag{15}$$

Let us now turn to the diagonalization problem. The associated transfer matrix is obtained as a graded trace of the monodromy matrix $\mathcal{T}(\lambda)$. The graded structure takes into account the fermionic degrees of freedom, and on the diagonal of $\mathcal{T}(\lambda)$ only $\hat{A}_{aa}(\lambda)$ contributes with a non-null Grassmann parity. Hence, the eigenvalue problem becomes

$$\left[B(\lambda) - \sum_{a=1}^{2} A_{aa}(\lambda) + D(\lambda)\right] |\Phi_n(\lambda_1, \ldots, \lambda_n)\rangle = \Lambda(\lambda, \{\lambda_i\})\,|\Phi_n(\lambda_1, \ldots, \lambda_n)\rangle\,. \tag{16}$$

In order to solve (16) we need the commutation rules between the diagonal and the creation operators. This is similar to solving a problem of quantum mechanics on the Fock space, analogously to the role of the Heisenberg algebra on the solution of the harmonic oscillator. In our case, the necessary commutation relations can be obtained by an appropriate manipulation of integrability condition (4). The procedure is rather cumbersome, and here we only list some of them which are fundamental for further discussion. They are given by

$$\begin{aligned}\hat{A}(\lambda) \otimes \boldsymbol{B}(\mu) = -\mathrm{i}\alpha_{1,9}(\lambda, \mu)[\boldsymbol{B}(\mu) \otimes \hat{A}(\lambda)] \cdot \hat{r}(\lambda, \mu) + \mathrm{i}\alpha_{5,9}(\lambda, \mu)\boldsymbol{B}(\lambda) \otimes \hat{A}(\mu) \\ +\{-\mathrm{i}\alpha_{10,7}(\lambda, \mu)[\boldsymbol{B}^*(\lambda)B(\mu) + \mathrm{i}\alpha_{5,9}(\lambda, \mu)F(\lambda)\boldsymbol{C}(\mu) \\ -\mathrm{i}\alpha_{2,9}(\lambda, \mu)F(\mu)\boldsymbol{C}(\lambda)]\} \otimes \boldsymbol{\xi}\end{aligned} \tag{17}$$

$$B(\lambda)\boldsymbol{B}(\mu) = \mathrm{i}\alpha_{2,9}(\mu, \lambda)\boldsymbol{B}(\mu)B(\lambda) - \mathrm{i}\alpha_{5,9}(\mu, \lambda)\boldsymbol{B}(\lambda)B(\mu) \tag{18}$$

$$\begin{aligned}D(\lambda)\boldsymbol{B}(\mu) = -\mathrm{i}\alpha_{8,7}(\lambda, \mu)\boldsymbol{B}(\mu)D(\lambda) + \alpha_{5,7}(\lambda, \mu)F(u)\boldsymbol{C}^*(\lambda) \\ -\alpha_{4,7}(\lambda, \mu)F(\lambda)\boldsymbol{C}^*(\mu) - \mathrm{i}\alpha_{10,7}(\lambda, \mu)\boldsymbol{\xi} \cdot [\boldsymbol{B}^*(\lambda) \otimes \hat{A}(\mu)].\end{aligned} \tag{19}$$

The eigenvalue $\Lambda(\lambda, \{\lambda_i\})$ can be calculated by keeping the terms proportional to the eigenstate $|\Phi_n(\lambda_1, \ldots, \lambda_n)\rangle$. For example, by using several times the first terms of the commutation relations (17)–(19) we find the following structure for eigenvalue $\Lambda(\lambda, \{\lambda_i\})$

$$\begin{aligned}\Lambda(\lambda, \{\lambda_i\}) = \left[\frac{a(\lambda)}{b(\lambda)}\mathrm{e}^{2h(\lambda)}\right]^L \prod_{i=1}^{n} \mathrm{i}\alpha_{2,9}(\lambda_i, \lambda) + \left[\frac{b(\lambda)}{a(\lambda)}\mathrm{e}^{2h(\lambda)}\right]^L \prod_{i=1}^{n} -\mathrm{i}\alpha_{8,7}(\lambda, \lambda_i) \\ - \prod_{i=1}^{n} -\mathrm{i}\alpha_{1,9}(\lambda, \lambda_i)\Lambda^{(1)}(\lambda, \{\lambda_j\}, \{\mu_j\})\end{aligned} \tag{20}$$

where $\Lambda^{(1)}(\lambda, \{\lambda_i\})$ is the eigenvalue of an auxiliary inhomogeneous problem related to the hidden six-vertex symmetry we have mentioned before. More precisely, such an auxiliary problem is defined by

$$\hat{r}_{b_1 d_1}^{c_1 a_1}(\lambda, \lambda_1)\hat{r}_{b_2 c_2}^{d_1 a_2}(\lambda, \lambda_2)\cdots\hat{r}_{b_n c_1}^{d_{n-1} a_n}(\lambda, \lambda_n)\mathcal{F}^{a_n \cdots a_1} = \Lambda^{(1)}(\lambda, \{\lambda_j\}, \{\mu_j\})\mathcal{F}^{b_n \cdots b_1}. \tag{21}$$

Fortunately such an additional eigenvalue problem can be solved using the well known results of Faddeev and coworkers [13]. New parameters $\{\mu_j\}$ are then introduced in order to perform the diagonalization problem (21). Here we just have to adapt their algebraic results in order to consider the six-vertex problem on an irregular lattice. Considering that this later eigenvalue problem has appeared in many different contexts in the literature [13–15], we just present our final results. First it is convenient to slightly generalize Shastry's parametrization [5] by introducing the new functions $z_\pm(x)$ as

$$z_-(x) = \frac{a(x)}{b(x)}\mathrm{e}^{2h(x)} \qquad z_+(x) = \frac{b(x)}{a(x)}\mathrm{e}^{2h(x)}. \tag{22}$$

In terms of functions $z_\pm(x)$ and the variables $\{\tilde{\mu_j}\}$ introduced in (11), we find that the eigenvalue (20) (modulo overall constant) can be written as

$$\begin{aligned}
\Lambda(\lambda, \{z_\pm(\lambda_i)\}, \{\tilde{\mu}_j\}) = {} & [z_-(\lambda)]^L \prod_{i=1}^{n} \frac{b(\lambda)}{a(\lambda)} \frac{1 + z_-(\lambda_i)/z_+(\lambda)}{1 - z_-(\lambda_i)/z_-(\lambda)} \\
& + [z_+(\lambda)]^L \prod_{i=1}^{n} \frac{b(\lambda)}{a(\lambda)} \frac{1 + z_-(\lambda_i)z_-(\lambda)}{1 - z_-(\lambda_i)z_+(\lambda)} \\
& - \prod_{i=1}^{n} \frac{b(\lambda)}{a(\lambda)} \frac{1 + z_-(\lambda_i)/z_+(\lambda)}{1 - z_-(\lambda_i)/z_-(\lambda)} \prod_{j=1}^{m} \frac{z_-(\lambda) - 1/z_-(\lambda) - \tilde{\mu}_j + U/2}{z_-(\lambda) - 1/z_-(\lambda) - \tilde{\mu}_j - U/2} \\
& - \prod_{i=1}^{n} \frac{b(\lambda)}{a(\lambda)} \frac{1 + z_-(\lambda_i)z_-(\lambda)}{1 - z_-(\lambda_i)z_+(\lambda)} \prod_{j=1}^{m} \frac{1/z_+(\lambda) - z_+(\lambda) - \tilde{\mu}_j - U/2}{1/z_+(\lambda) - z_+(\lambda) - \tilde{\mu}_j + U/2}.
\end{aligned} \tag{23}$$

Analogously, in order to cancel the unwanted terms, it is possible to show that the nested Bethe ansatz equations constraining the numbers $\{\lambda_i\}$,$\{\tilde{\mu}_j\}$ are then given by

$$[z_-(\lambda_k)]^L = \prod_{j=1}^{m} \frac{z_-(\lambda_k) - 1/z_-(\lambda_k) - \tilde{\mu}_j + U/2}{z_-(\lambda_k) - 1/z_-(\lambda_k) - \tilde{\mu}_j - U/2} \qquad k = 1, \ldots, n \tag{24}$$

$$\prod_{k=1}^{n} \frac{z_-(\lambda_k) - 1/z_-(\lambda_k) - \tilde{\mu}_l - U/2}{z_-(\lambda_k) - 1/z_-(\lambda_k) - \tilde{\mu}_l + U/2} = -\prod_{j=1}^{n} \frac{\tilde{\mu}_l - \tilde{\mu}_j + U}{\tilde{\mu}_l - \tilde{\mu}_j - U} \qquad l = 1, \ldots, m. \tag{25}$$

To check the consistency of our results (23)–(25) one has to verify that $\Lambda(\lambda, \{z_\pm(\lambda_i)\}, \{\tilde{\mu}_j\})$ is free of poles for finite values of λ. In fact, the null residue condition on both direct ($z_-(\lambda)$) and crossed ($z_+(\lambda)$) poles lead us to the Bethe conditions (24) and (25). A possible physical application of the eigenvalue result (23) is probably concerned with the finite-temperature properties of the one-dimensional Hubbard model [16]. This is connected to the recent developments of new powerful methods to deal with finite-size effects in integrable models [17, 18]. These techniques depend much on the diagonalization of the quantum transfer matrix (rather the one-dimensional Hamiltonian), a problem which we have managed to solve in this letter. Lastly, it is also possible to rewrite the nested Bethe ansatz equations (24) and (25) in terms of the original form presented by Lieb and Wu [1]. In this case, one just needs to change $\tilde{\mu}_j \to 2i\tilde{\mu}_j$ and relate the variables λ_k with the lattice momenta p_k [5] by the relation $z_-(\lambda_k) = \mathrm{e}^{\mathrm{i}p_k}$.

We would like to conclude this letter with the following comments. The eigenvalue (23) is almost the one conjectured by Shastry in [5]. They differ by important phase factors, which are not easily obtained by using only phenomenological arguments. Our result (23)–(25) is connected with periodic boundary conditions, while that conjectured by Shastry is related to rather peculiar (sector-dependent) toroidal boundary conditions. The method we have presented here is easily extended for a more general inhomogeneous model [5, 10]. We expect that the only change in the Bethe ansatz equations (24) and (25) will be on the terms proportional to the power of L. We plan to discuss these results in a more detailed version of this letter [19]. Finally, some extra remarks are now in order. It is possible to show, from the commutation rules between the 'dual' field $\boldsymbol{B}^*(\lambda)$ and $F(\lambda)$, that a second $SU(2)$ six-vertex hidden symmetry is also present [19]. Thus, the two six-vertex structure are tied up by the same field $F(\lambda)$. This resembles much the constrain leading to the $SO(4)$ symmetry of the Hubbard chain [20]. This is known to be of relevance for the Bethe ansatz completeness [21], for the classification of the elementary excitations [22], and can also play an important role in the computation of correlation functions [15].

The work of PBR is supported by Fapesp. MJM is partially supported by Cnpq and Fapesp.

Appendix

The structure of the R-matrix [4–6] is

$$R(\lambda,\mu)=\begin{pmatrix}
\alpha_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \alpha_5 & 0 & 0 & -\mathrm{i}\alpha_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}\alpha_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \alpha_4 & 0 & 0 & -\mathrm{i}\alpha_{10} & 0 & 0 & \mathrm{i}\alpha_{10} & 0 & 0 & \alpha_7 & 0 & 0 & 0\\
0 & -\mathrm{i}\alpha_8 & 0 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \mathrm{i}\alpha_{10} & 0 & 0 & \alpha_3 & 0 & 0 & -\alpha_6 & 0 & 0 & -\mathrm{i}\alpha_{10} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}\alpha_8 & 0 & 0\\
0 & 0 & -\mathrm{i}\alpha_8 & 0 & 0 & 0 & 0 & 0 & \alpha_5 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -\mathrm{i}\alpha_{10} & 0 & 0 & -\alpha_6 & 0 & 0 & \alpha_3 & 0 & 0 & \mathrm{i}\alpha_{10} & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_5 & 0 & 0 & -\mathrm{i}\alpha_8 & 0\\
0 & 0 & 0 & \alpha_7 & 0 & 0 & \mathrm{i}\alpha_{10} & 0 & 0 & -\mathrm{i}\alpha_{10} & 0 & 0 & \alpha_4 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}\alpha_9 & 0 & 0 & 0 & 0 & 0 & \alpha_5 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\mathrm{i}\alpha_9 & 0 & 0 & \alpha_5 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \alpha_2
\end{pmatrix} \tag{A.1}$$

where the weights $\alpha_i(\lambda,\mu)$ (normalized by $\alpha_5(\lambda,\mu)$) are given by

$$\alpha_1(\lambda,\mu)=\{\mathrm{e}^{[h(\mu)-h(\lambda)]}a(\lambda)a(\mu)+\mathrm{e}^{-[h(\mu)-h(\lambda)]}b(\lambda)b(\mu)\}\alpha_5(\lambda,\mu) \tag{A.2}$$

$$\alpha_2(\lambda,\mu)=\{\mathrm{e}^{-[h(\mu)-h(\lambda)]}a(\lambda)a(\mu)+\mathrm{e}^{[h(\mu)-h(\lambda)]}b(\lambda)b(\mu)\}\alpha_5(\lambda,\mu) \tag{A.3}$$

$$\alpha_3(\lambda,\mu)=\frac{\mathrm{e}^{[h(\mu)+h(\lambda)]}a(\lambda)b(\mu)+\mathrm{e}^{-[h(\mu)+h(\lambda)]}b(\lambda)a(\mu)}{a(\lambda)b(\lambda)+a(\mu)b(\mu)} \times\left\{\frac{\cosh[h(\mu)-h(\lambda)]}{\cosh[h(\mu)+h(\lambda)]}\right\}\alpha_5(\lambda,\mu) \tag{A.4}$$

$$\alpha_4(\lambda,\mu)=\frac{\mathrm{e}^{-[h(\mu)+h(\lambda)]}a(\lambda)b(\mu)+\mathrm{e}^{[h(\mu)+h(\lambda)]}b(\lambda)a(\mu)}{a(\lambda)b(\lambda)+a(\mu)b(\mu)} \times\left\{\frac{\cosh(h(\mu)-h(\lambda))}{\cosh(h(\mu)+h(\lambda))}\right\}\alpha_5(\lambda,\mu) \tag{A.5}$$

$$\alpha_6(\lambda,\mu)=\left\{\frac{\mathrm{e}^{[h(\mu)+h(\lambda)]}a(\lambda)b(\mu)-\mathrm{e}^{-[h(\mu)+h(\lambda)]}b(\lambda)a(\mu)}{a(\lambda)b(\lambda)+a(\mu)b(\mu)}\right\}$$
$$\times[b^2(\mu)-b^2(\lambda)]\frac{\cosh[h(\mu)-h(\lambda)]}{\cosh[h(\mu)+h(\lambda)]}\alpha_5(\lambda,\mu) \qquad \text{(A.6)}$$

$$\alpha_7(\lambda,\mu)=\left\{\frac{-\mathrm{e}^{-[h(\mu)+h(\lambda)]}a(\lambda)b(\mu)+\mathrm{e}^{[h(\mu)+h(\lambda)]}b(\lambda)a(\mu)}{a(\lambda)b(\lambda)+a(\mu)b(\mu)}\right\}$$
$$\times[b^2(\mu)-b^2(\lambda)]\frac{\cosh[h(\mu)-h(\lambda)]}{\cosh[h(\mu)+h(\lambda)]}\alpha_5(\lambda,\mu) \qquad \text{(A.7)}$$

$$\alpha_8(\lambda,\mu)=\{\mathrm{e}^{[h(\mu)-h(\lambda)]}a(\lambda)b(\mu)-\mathrm{e}^{-[h(\mu)-h(\lambda)]}b(\lambda)a(\mu)\}\alpha_5(\lambda,\mu) \qquad \text{(A.8)}$$

$$\alpha_9(\lambda,\mu)=\{-\mathrm{e}^{-[h(\mu)-h(\lambda)]}a(\lambda)b(\mu)+\mathrm{e}^{[h(\mu)-h(\lambda)]}b(\lambda)a(\mu)\}\alpha_5(\lambda,\mu) \qquad \text{(A.9)}$$

$$\alpha_{10}(\lambda,\mu)=\frac{b^2(\mu)-b^2(\lambda)}{a(\lambda)b(\lambda)+a(\mu)b(\mu)}\left\{\frac{\cosh[h(\mu)-h(\lambda)]}{\cosh[h(\mu)+h(\lambda)]}\right\}\alpha_5(\lambda,\mu). \qquad \text{(A.10)}$$

We note that we have used the original Shastry's Boltzmann weights [4, 5] together with the grading modifications of Wadati *et al* [6]. Moreover, the six-vertex parameters $a(\lambda)$ and $b(\lambda)$ satistfy the free-fermion condition $a(\lambda)^2+b(\lambda)^2=1$. We also list some important identities between the Boltzmann weights [6]

$$\alpha_3(\lambda,\mu)=\alpha_1(\lambda,\mu)+\alpha_6(\lambda,\mu);\,\alpha_4(\lambda,\mu)+\alpha_7(\lambda,\mu)=\alpha_2(\lambda,\mu) \qquad \text{(A.11)}$$

$$\alpha_2(\lambda,\mu)\alpha_1(\lambda,\mu)-\alpha_9(\lambda,\mu)\alpha_8(\lambda,\mu)=\alpha_4(\lambda,\mu)\alpha_3(\lambda,\mu)-\alpha_{10}^2(\lambda,\mu)=\alpha_5^2(\lambda,\mu) \qquad \text{(A.12)}$$

$$\alpha_2(\lambda,\mu)\alpha_3(\lambda,\mu)+\alpha_4(\lambda,\mu)\alpha_1(\lambda,\mu)=2\alpha_5^2(\lambda,\mu). \qquad \text{(A.13)}$$

References

[1] Lieb E H and Wu F Y 1968 *Phys. Rev. Lett.* **20** 1445
[2] For a review see, Sutherland B 1985 *Lecture Notes on Physics 242* ed B S Shastry, S S Jha and V Singh
Andrei N 1992 *Integrable Model in Condensed Matter Physics (ICTP Summer Course, 1992)* (Singapore: World Scientific)
[3] Shastry B S 1986 *Phys. Rev. Lett.* **56** 1529
[4] Shastry B S 1986 *Phys. Rev. Lett.* **56** 2453
[5] Shastry B S 1988 *J. Stat. Phys.* **30** 57
[6] Wadati M, Olmedilla E and Akutsu Y 1987 *J. Phys. Soc. Japan* **36** 1340
Wadati M and Olmedilla E 1987 *J. Phys. Soc. Japan* **36** 2298
Wadati M, Olmedilla E and Akutsu Y 1988 *Phys. Rev. Lett.* **60** 1595
[7] Bariev R Z 1990 *Theor. Math. Fiz* **82** 313
[8] Bariev R Z 1981 *Theor. Math. Fiz* **49** 261
Truong T T and Schotte K D 1984 *Nucl. Phys.* B **230** 1
[9] Uglov D B and Korepin V E 1994 *Phys. Lett.* **190A** 238
[10] Shiroishi M and Wadati M 1995 *J. Phys. Soc. Japan* **64** 57
[11] Grabowski M P and Mathieu P 1995 *Ann. Phys.* **243** 299
Grosse H 1989 *Lett. Math. Phys.* **18** 151
[12] Kulish P and Sklyanin E 1982 *J. Sov. Math.* **19** 1596
[13] Faddeev L D 1984 *Recent Advances in Field Theory and Statistical Mechanics (Les Houches, 1982)* (Amsterdam: Elsevier)
Takhtajan L A and Faddeev L D 1979 *Rus. Math. Sur.* **34** 11
[14] de Vega H J 1989 *Int. Mod. Phys.* A **4** 2371
Takhtajan L A 1985 *Exactly Solvable Problems in Condensed Matter and Relativistic Field Theory (Lecture Notes in Physics 242)* ed B S Shastry, S S Jha and V Singh p 175
[15] Korepin V E, Izergin G and Bogoliubov N M 1992 *Quantum Inverse Scattering Method, Correlation Functions and Algebraic Bethe Ansatz* (Cambridge: Cambridge University Press)

[16] Klümper A and Bariev R Z 1996 *Nucl. Phys.* B **458** 623
[17] Pearce P A and Klümper A 1991 *Phys. Rev. Lett.* **66** 623
Klümper A and Pearce P A 1992 *Physica* A **183** 304
[18] Destri C and de Vega H J 1992 *Phys. Rev. Lett.* **69** 2313
Destri C and de Vega H J 1995 *Nucl. Phys.* B **438** 413
[19] Martins M J and Ramos P B to be published
[20] Heilmann O J and Lieb E H 1971 *Ann. Acad. Sci., NY* **172** 583
Yang C N 1989 *Phys. Rev. Lett.* **63** 2144
Pernici M 1990 *Europhys. Lett.* **12** 75
[21] Essler F H L, Korepin V E and Schoutens K 1991 *Phys. Rev. Lett.* **67** 3848
[22] Essler F H L and Korepin V E 1994 *Phys. Rev. Lett.* **72** 908